

ON THE CONSTANT IN A TRANSFERENCE INEQUALITY FOR THE VECTOR-VALUED FOURIER TRANSFORM

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ABSTRACT. The standard proof of the equivalence of Fourier type on \mathbb{R}^d and on the torus \mathbb{T}^d is usually stated in terms of an implicit constant, defined as the minimum of a sum of powers of sinc functions. In this note we compute this minimum explicitly.

1. INTRODUCTION

The motivation of this paper comes from a well-known transference result for the vector-valued Fourier transform. Let X be a complex Banach space. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d; X)$ is defined by

$$\mathcal{F}_{\mathbb{R}^d} f(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^d.$$

Likewise, the *Fourier transform* of a function $f \in L^1(\mathbb{T}^d; X)$ is defined by

$$\mathcal{F}_{\mathbb{T}^d} f(k) := \int_{\mathbb{T}^d} e^{-2\pi i k \cdot t} f(t) dt, \quad k \in \mathbb{Z}^d.$$

Proposition 1. *Let X be a complex Banach space, fix $d \geq 1$ and $p \in (1, 2]$, and let $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions are equivalent:*

- (i) $\mathcal{F}_{\mathbb{R}^d}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ into $L^q(\mathbb{R}^d; X)$;
- (ii) $\mathcal{F}_{\mathbb{T}^d}$ extends to a bounded operator from $L^p(\mathbb{T}^d; X)$ into $\ell^q(\mathbb{Z}^d; X)$.

In this situation, denoting the norms of these extensions by $\varphi_{p,X}(\mathbb{R}^d)$ and $\varphi_{p,X}(\mathbb{T}^d)$, we have

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq C_q^{-d/q} \varphi_{p,X}(\mathbb{R}^d),$$

where C_q is the global minimum of the periodic function

$$x \mapsto \sum_{m \in \mathbb{Z}} \left| \frac{\sin(\pi(x+m))}{\pi(x+m)} \right|^q, \quad x \in \mathbb{R}.$$

This function, as well as several others considered below, have removable singularities. It is understood that we will always be working with their unique continuous extensions.

A complex Banach space X which has the equivalent properties (i) and (ii) is said to have *Fourier type p* ; this notion has been introduced in [5]. Proposition 1 goes back to [4]; in its stated form the result can be found in [2, 3]. Related results may be found in [1]. These references do not comment on the location of the global minimum. A quick computer plot (see Figure 1) suggests that the minimum is taken in the points $\frac{1}{2} + \mathbb{Z}$. To actually *prove* this turns out to be surprisingly difficult. This is the modest objective of the present note:

Proposition 2. *For every real number $r \geq 1$, the function $f_r : [0, 1] \rightarrow \mathbb{R}$ defined by*

$$f_r(x) := \sum_{m \in \mathbb{Z}} \left| \frac{\sin(\pi(x+m))}{\pi(x+m)} \right|^{2r}, \quad x \in [0, 1],$$

has a global minimum at $x = \frac{1}{2}$.

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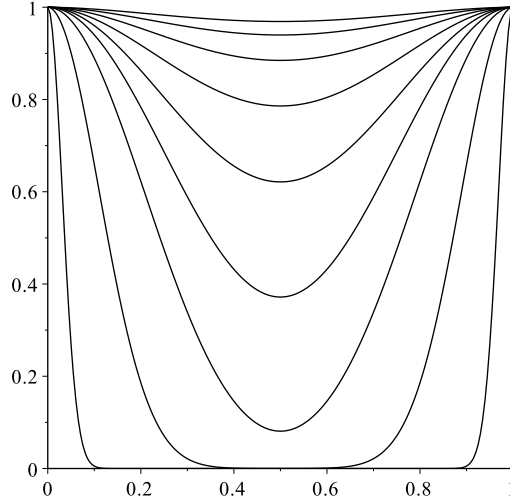


FIGURE 1. A plot of f_r , where $r = 1.02^k$ for $k = 1, 2, 4, \dots, 256$.

Our proof has developed essentially by trial and error. We believe it is perfectly possible that a truly pedestrian proof can be given, but we failed to find one despite many hours of efforts.

As a consequence of Proposition 2 we obtain the explicit estimate

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq \frac{\pi^d}{(2(2^q - 1)\zeta(q))^{d/q}} \cdot \varphi_{p,X}(\mathbb{R}^d),$$

noting that

$$\sum_{m \in \mathbb{Z}} \frac{1}{|\frac{1}{2} + m|^q} = 2(2^q - 1)\zeta(q).$$

For even integers $q = 2n$, the constant on the right-hand side may be evaluated explicitly in terms of the Bernoulli numbers. To further estimate this constant, recall that for any $x \in \ell^2(\mathbb{Z})$ the function $q \mapsto \|x\|_q := (\sum_{m \in \mathbb{Z}} |x_m|^q)^{1/q}$ is decreasing on $[2, \infty)$ and $\lim_{q \rightarrow \infty} \|x\|_q = \sup_{i \in \mathbb{Z}} |x_i|$. Taking $x_m := |\frac{1}{2} + m|^{-1}$ we find $(\sum_{m \in \mathbb{Z}} |\frac{1}{2} + m|^{-q})^{1/q} \geq 2$ for every $q \geq 2$, and hence in particular

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq (\frac{1}{2}\pi)^d \varphi_{p,X}(\mathbb{R}^d).$$

2. THE MAIN RESULT

The proof of the proposition is based on the following lemmas. The main idea is contained in the first lemma.

Lemma 3. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing convex function, and let $x_1, \dots, x_n \in \mathbb{R}_+$ and $y_1, \dots, y_n \in \mathbb{R}_+$ be such that*

- (i) $x_1 + \dots + x_n \geq y_1 + \dots + y_n$;
- (ii) *there exists $t \in \mathbb{R}_+$ such that*
 - $x_i \leq y_i$ if $y_i < t$;
 - $x_i \geq y_i$ if $y_i \geq t$.

Then $g(x_1) + \dots + g(x_n) \geq g(y_1) + \dots + g(y_n)$.

Proof. We will prove the lemma by induction on n . The case $n = 1$ is clear: $x_1 \geq y_1$ implies that $g(x_1) \geq g(y_1)$ since g is non-decreasing. Suppose now that the lemma has been proved for $n = 1, \dots, m-1$.

If $x_i = y_i$ for some index $1 \leq i \leq m$, then we may remove x_i and y_i and apply the induction hypothesis.

If $x_i \geq y_i$ for every index $1 \leq i \leq m$, then again the result is immediate since g is non-decreasing. Therefore, we may assume that $x_i < y_i$ for some index $1 \leq i \leq m$. Then, by the first condition in the lemma, there is also an index j for which $x_j > y_j$. By the second condition in the lemma we then have $x_i < y_i < t \leq y_j < x_j$.

Let $\epsilon := \min(y_i - x_i, x_j - y_j)$ and define $x'_i := x_i + \epsilon$, $x'_j := x_j - \epsilon$, and $x'_k := x_k$ for all other indices. Then $x'_1, \dots, x'_m, y_1, \dots, y_m$ satisfy the conditions in the lemma (with the same t) and $x'_i = y_i$ or $x'_j = y_j$. Hence, by the induction hypothesis, we have

$$g(x'_1) + \dots + g(x'_m) \geq g(y_1) + \dots + g(y_m). \quad (1)$$

Since $x_i \leq x'_i \leq x'_j \leq x_j$, we can write $x'_i = \lambda x_i + (1 - \lambda)x_j$ for some $\lambda \in [0, 1]$. Since $x'_j = x_i + x_j - x'_i$, we have $x'_j = (1 - \lambda)x_i + \lambda x_j$. By the convexity of g it follows that

$$g(x'_i) + g(x'_j) \leq (\lambda g(x_i) + (1 - \lambda)g(x_j)) + ((1 - \lambda)g(x_i) + \lambda g(x_j)) = g(x_i) + g(x_j). \quad (2)$$

Combining inequalities (1) and (2) we obtain the lemma for $n = m$, thus completing the induction step. \square

In order to apply this lemma we need a number of technical facts. The first (cf. [2, (6.14)]) is elementary and is left as an exercise.

Lemma 4. $f_1(x) = 1$ for all $x \in [0, 1]$.

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$h(x) := \text{sinc}^2(\pi x) = \left(\frac{\sin(\pi x)}{\pi x} \right)^2, \quad x \in \mathbb{R}.$$

Lemma 5. Let $r \geq 1$. The following assertions hold on the interval $[0, 1]$:

- (i) the function $h(x) + h(x - 1)$ has a global minimum at $x = \frac{1}{2}$;
- (ii) for all $m = 1, 2, 3, \dots$, $h(x + m) + h(x - (m + 1))$ has a global maximum at $x = \frac{1}{2}$;
- (iii) the function

$$h(x) + h(x - 1) - (h(x)^r + h(x - 1)^r)^{1/r}$$

has a global maximum at $x = \frac{1}{2}$;

- (iv) for all $m = 1, 2, 3, \dots$ and $r \geq 1$,

$$(h(x + m) + h(x - (m + 1)))^r - h(x + m)^r - h(x - (m + 1))^r$$

has a global maximum at $x = \frac{1}{2}$.

Assuming the lemmas for the moment, let us first show how the proposition can be deduced from them.

Proof of Proposition 2. Fix $r \geq 1$ and set, for $x \in [0, 1]$,

$$s_m(x) := h(x + m) + h(x - (m + 1)) \quad (m = 0, 1, 2, \dots)$$

and

$$\tilde{s}_0(x) := ((h(x))^r + (h(x - 1))^r)^{1/r}.$$

In view of part (iv) of Lemma 5 it suffices to prove that

$$\tilde{s}_0^r + s_1^r + s_2^r + \dots$$

has a global minimum at $x = \frac{1}{2}$.

Fix an arbitrary $x \in [0, 1]$ and set

$$x_m := s_m(x), \quad y_m := s_m\left(\frac{1}{2}\right) \quad (m = 0, 1, 2, \dots)$$

and

$$\tilde{x}_0 := ((h(x))^r + h(x - 1)^r)^{1/r}, \quad \tilde{y}_0 := ((h(\frac{1}{2}))^r + h(-\frac{1}{2})^r)^{1/r}.$$

In view of parts (i) and (ii) of Lemma 5 we have

$$x_0 \geq y_0, \quad x_i \leq y_i \quad (i = 1, 2, \dots) \quad (3)$$

Lemma 4 implies

$$x_0 + x_1 + x_2 + \cdots = y_0 + y_1 + y_2 + \cdots \quad (4)$$

By (3) and (4),

$$x_0 + x_1 + \cdots + x_n \geq y_0 + y_1 + \cdots + y_n \quad (n = 0, 1, 2, \dots) \quad (5)$$

Part (iii) of Lemma 5 implies

$$\tilde{x}_0 - x_0 \geq \tilde{y}_0 - y_0. \quad (6)$$

By (5) and (6),

$$\tilde{x}_0 + x_1 + \cdots + x_n \geq \tilde{y}_0 + y_1 + \cdots + y_n \quad (n = 0, 1, 2, \dots) \quad (7)$$

Finally, by (3) and (6),

$$\tilde{x}_0 \geq \tilde{y}_0. \quad (8)$$

A simple calculation shows that $\tilde{y}_0 > \frac{4}{\pi^2}$ and $y_i < \frac{4}{\pi^2}$ for $i = 1, 2, \dots$. Taking $t = \frac{4}{\pi^2}$ in Lemma 3 and $g(x) := x^r$ now implies, by virtue of (3), (7), and (8), that

$$\tilde{x}_0^r + x_1^r + \cdots + x_n^r \geq \tilde{y}_0^r + y_1^r + \cdots + y_n^r$$

holds for every n . Taking limits for $n \rightarrow \infty$ completes the proof. \square

3. PROOF OF LEMMA 5

This section is devoted to the proof of Lemma 5, which is based on the following observations:

Lemma 6. *On the interval $[0, 1]$:*

- (i) $\frac{\cos(\frac{1}{2}\pi x)}{1 - x^2}$ takes a global maximum at $x = 0$;
- (ii) $\frac{(x^2 + 1)\cos^2(\frac{1}{2}\pi x)}{(1 - x^2)^2}$ takes a global minimum at $x = 0$.

Proof. We start by showing that

$$\sqrt{2}\sin(\tfrac{1}{4}\pi x) \geq x \quad \text{for all } x \in [0, 1]. \quad (9)$$

To this end, consider the function $f(x) := \sqrt{2}\sin(\frac{1}{4}\pi x) - x$. Observe that $f'(x) = \frac{\pi\sqrt{2}}{4}\cos(\frac{1}{4}\pi x) - 1$ is decreasing on $[0, 1]$, hence f is concave. Since $f(0) = f(1) = 0$ this implies that $f(x) \geq 0$ for $x \in [0, 1]$, which proves the claim.

(i): The value at $x = 0$ of the given function equals 1, so it suffices to show that $\cos(\frac{1}{2}\pi x) \leq 1 - x^2$ for all $x \in [0, 1]$. This follows from the double-angle formula for cosine and (9):

$$\cos(\tfrac{1}{2}\pi x) = 1 - 2\sin^2(\tfrac{1}{4}\pi x) \leq 1 - x^2.$$

(ii): The given function has value 1 at $x = 0$, hence it suffices to show that for all $x \in [0, 1]$,

$$(x^2 + 1)\cos^2(\tfrac{1}{2}\pi x) \geq (1 - x^2)^2.$$

On the interval $[\frac{1}{2}, 1]$ we substitute $x = 1 - y$. We then must prove that for $y \in [0, \frac{1}{2}]$,

$$(2 - 2y + y^2)\sin^2(\tfrac{1}{2}\pi y) \geq (2y - y^2)^2.$$

Since $2y \in [0, 1]$, we can use (9) to obtain $\sqrt{2}\sin(\frac{1}{4}\pi \cdot 2y) \geq 2y$, and hence $\sin^2(\frac{1}{2}\pi y) \geq 2y^2$. This implies that

$$(2 - 2y + y^2)\sin^2(\tfrac{1}{2}\pi y) \geq (2 - 2y + y^2)(2y^2) = (y^2 + (2 - y)^2)y^2 \geq (2 - y)^2y^2 = (2y - y^2)^2,$$

which concludes the proof on the interval $[\frac{1}{2}, 1]$.

For $x \in [0, \frac{1}{2}]$ we have

$$\begin{aligned}
(x^2 + 1) \cos^2(\tfrac{1}{2}\pi x) &\geq (x^2 + 1) \left(1 - \frac{\pi^2}{8}x^2\right)^2 \\
&= (x^2 + 1) \left(1 - \frac{\pi^2}{4}x^2 + \frac{\pi^4}{64}x^4\right) \\
&\geq 1 + \left(1 - \frac{\pi^2}{4}\right)x^2 + \left(\frac{\pi^4}{64} - \frac{\pi^2}{4}\right)x^4 \\
&= 1 + \left(1 - \frac{\pi^2}{4}\right)x^2 + \left(\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1\right)x^4 + x^4 \\
&\geq 1 + \left[\left(1 - \frac{\pi^2}{4}\right) + \frac{1}{4}\left(\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1\right)\right]x^2 + x^4 \\
&\geq 1 - 2x^2 + x^4 \\
&= (1 - x^2)^2,
\end{aligned}$$

noting that $\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1 < 0$ and $(1 - \frac{\pi^2}{4}) + \frac{1}{4}(\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1) \approx -1.9537471 \dots > -2$ \square

Proof of Lemma 5. (i): We have

$$h(x) + h(x-1) = \frac{\sin^2(\pi x)}{\pi^2 x^2} + \frac{\sin^2(\pi x)}{\pi^2 (x-1)^2} = \frac{(2x^2 - 2x + 1) \sin^2(\pi x)}{\pi^2 x^2 (x-1)^2} =: g(x).$$

We must show that

$$f(x) := g\left(x + \frac{1}{2}\right) = \frac{8}{\pi^2} \frac{4x^2 + 1}{(4x^2 - 1)^2} \cos^2(\pi x)$$

has a global minimum in $x = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. But this follows from Lemma 6 and the fact that f is even.

(ii): For $m = 1, 2, 3, \dots$ we have

$$h(x+m) + h(x-(m+1)) = \frac{[2x^2 - 2x + (m+1)^2 + m^2] \sin^2(\pi x)}{\pi^2 [(x+m)^2 (x-(m+1))^2]} =: g_m(x).$$

We must show that

$$f_m(x) := g_m\left(x + \frac{1}{2}\right) = \frac{8}{\pi^2} \frac{4x^2 + 4m^2 + 4m + 1}{[(4x^2 - (2m+1)^2)^2]} \cos^2(\pi x)$$

has a global maximum in $x = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. For this, it suffices to check that the functions

$$\frac{4x^2 + 1}{(4x^2 - M^2)^2} \cos^2(\pi x) \quad \text{and} \quad \frac{1}{(4x^2 - M^2)^2} \cos^2(\pi x)$$

are decreasing on $[0, \frac{1}{2}]$ for each $M \geq 3$, or equivalently, that

$$\frac{\sqrt{x^2 + 1}}{M^2 - x^2} \cos(\tfrac{1}{2}\pi x) \quad \text{and} \quad \frac{1}{M^2 - x^2} \cos(\tfrac{1}{2}\pi x)$$

are decreasing on $[0, 1]$ for each $M \geq 3$. It suffices to prove this for the first function, since this will immediately imply the result for the second function.

Straightforward algebra shows that the derivative of the function

$$\psi_M(x) := \frac{\sqrt{x^2 + 1}}{M^2 - x^2} \cos(\tfrac{1}{2}\pi x)$$

has a zero at x if and only if

$$2x(x^2 + 2 + M^2) \cos(\tfrac{1}{2}\pi x) = \pi(M^2 - x^4 + (M^2 - 1)x^2) \sin(\tfrac{1}{2}\pi x).$$

But,

$$2x(M^2 + 2 + x^2) \cos(\tfrac{1}{2}\pi x) \leq 2x(M^2 + 2 + x^2)$$

and, since $0 \leq x \leq 1$,

$$\pi(M^2 - x^4 + (M^2 - 1)x^2)x \leq \pi(M^2 - x^4 + (M^2 - 1)x^2) \sin(\tfrac{1}{2}\pi x),$$

while also, using that $M \geq 3$ and $0 \leq x \leq 1$,

$2(M^2 + 2 + x^2) \leq 2(M^2 + 2 + (M^2 - 1)x^2) < \pi(M^2 - 1 + (M^2 - 1)x^2) \leq \pi(M^2 - x^4 + (M^2 - 1)x^2)$
since $2(M^2 + 2) < \pi(M^2 - 1)$ for $M \geq 3$. It follows that the derivative of ψ_M has no zeros on $(0, 1]$, and then from

$$\psi_M(0) = \frac{1}{M^2} > 0 = \psi_M(1)$$

it follows that ψ_M is decreasing on $[0, 1]$.

(iii): Proceeding as in (i), we have

$$\begin{aligned} & h(x) + h(x-1) - ((h(x))^r + (h(x-1))^r)^{1/r} \\ &= \frac{1}{\pi^2} \left[\frac{1}{x^2} + \frac{1}{(1-x)^2} - \left(\frac{1}{x^{2r}} + \frac{1}{(1-x)^{2r}} \right)^{1/r} \right] \sin^2(\pi x) =: g(x). \end{aligned}$$

We must show that

$$f(x) := g\left(\frac{1}{2} + x\right) = \frac{1}{\pi^2} \left[\left(\frac{1}{2} - x\right)^2 + \left(\frac{1}{2} + x\right)^2 - \left(\left(\frac{1}{2} - x\right)^{2r} + \left(\frac{1}{2} + x\right)^{2r}\right)^{1/r} \right] \frac{\cos^2(\pi x)}{\left(\frac{1}{4} - x^2\right)^2}$$

has a global maximum in $x = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The function f is even, and by Lemma 6, $\cos^2(\pi x)/(\frac{1}{4} - x^2)^2$ takes its maximum at $x = 0$. It thus remains to show that on the interval $[0, \frac{1}{2}]$ the function

$$\phi_r(x) := \left(\frac{1}{2} - x\right)^2 + \left(\frac{1}{2} + x\right)^2 - \left(\left(\frac{1}{2} - x\right)^{2r} + \left(\frac{1}{2} + x\right)^{2r}\right)^{1/r}$$

is decreasing on $[0, \frac{1}{2}]$. The derivative of this function equals

$$\phi'_r(x) = 4x - 2\left(\left(\frac{1}{2} - x\right)^{2r} + \left(\frac{1}{2} + x\right)^{2r}\right)^{1/r-1} \left(\left(\frac{1}{2} + x\right)^{2r-1} - \left(\frac{1}{2} - x\right)^{2r-1}\right).$$

To show that $\phi'_r(x) \leq 0$ we must show that

$$\left(\left(\frac{1}{2} + x\right)^{2r} + \left(\frac{1}{2} - x\right)^{2r}\right)^{1/r-1} \left(\left(\frac{1}{2} + x\right)^{2r-1} - \left(\frac{1}{2} - x\right)^{2r-1}\right) \geq 2x$$

for $x \in [0, \frac{1}{2}]$, or, after substituting $a = \frac{1}{2} + x$ and $b = \frac{1}{2} - x$, that

$$a^{2r-1} - b^{2r-1} \geq (a-b)(a^{2r} + b^{2r})^{1-1/r}$$

for all $a \in [\frac{1}{2}, 1]$. In view of

$$\begin{aligned} (a^{2r} + b^{2r})^{1-1/r} &= \left[(a^{2r} + b^{2r})^{1/(2r)} \right]^{2r-2} \\ &\leq \left[(a^{2r-1} + b^{2r-1})^{1/(2r-1)} \right]^{2r-2} = (a^{2r-1} + b^{2r-1})^{1-1/(2r-1)}, \end{aligned}$$

with $p := 2r - 1$ it suffices to show that

$$a^p - b^p \geq (a-b)(a^p + b^p)^{1-1/p}$$

for all $a \geq b \geq 0$. We can further simplify this upon dividing both sides by b^p . In the new variable $x = a/b$ we then have to prove that

$$x^p - 1 \geq (x-1)(x^p + 1)^{1-1/p}$$

for all $x \geq 1$.

Using that $(1+y)^\alpha \leq 1 + \alpha y$ for $y \geq 0$ and $0 \leq \alpha \leq 1$, we have

$$(x-1)(x^p + 1)^{1-1/p} = (x^p - x^{p-1})(1 + x^{-p})^{1-1/p} \leq (x^p - x^{p-1})[1 + (1 - \frac{1}{p})x^{-p}].$$

Therefore it remains to prove that for $x \geq 1$ and $p \geq 1$ we have

$$x^p - 1 \geq (x^p - x^{p-1})[1 + (1 - \frac{1}{p})x^{-p}],$$

or, multiplying both sides with x , that

$$x^{p+1} - x \geq x^{p+1} - x^p + (1 - \frac{1}{p})(x-1).$$

that is, we must show that

$$f_p(x) := x^p \geq x + (1 - \frac{1}{p})(x - 1) =: g_p(x).$$

Now

$$f'_p(x) = px^{p-1}, \quad g'_p(x) = 2 - \frac{1}{p}.$$

It follows that $f'_p(x) \geq g'_p(x) \geq 0$ for $x \geq 1$, since $p \geq 2 - \frac{1}{p}$ (multiply both sides by p). Together with $f_p(1) = g_p(1)$ it follows that $f_p(x) \geq g_p(x)$ for $x \geq 1$ and $p \geq 1$. This concludes the proof of (iii).

(iv): Fix $m \geq 1$. For $x \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$\begin{aligned} & (h(x + \frac{1}{2} + m) + h(x + \frac{1}{2} - (m + 1)))^r - h(x + \frac{1}{2} + m)^r - h(x + \frac{1}{2} - (m + 1))^r \\ &= \left[\left(\frac{1}{(x + (m + \frac{1}{2}))^2} + \frac{1}{(x - (m + \frac{1}{2}))^2} \right)^r - \left(\frac{1}{(x + (m + \frac{1}{2}))^2} \right)^r - \left(\frac{1}{(x - (m + \frac{1}{2}))^2} \right)^r \right] \\ & \quad \times \pi^{-2r} (\cos^2(\pi x))^r. \end{aligned}$$

We must show that this function has a global maximum on $[-\frac{1}{2}, \frac{1}{2}]$ at $x = 0$. Since by Lemma 6 $\cos(\pi x)/(1 - 4x^2)$ has a global maximum at $x = 0$, it suffices to prove that

$$[((a + x)^{-2} + (a - x)^{-2})^r - (a + x)^{-2r} - (a - x)^{-2r}] \cdot (1 - 4x^2)^{2r}$$

has a global maximum at $x = 0$, where we have written $a := m + \frac{1}{2} \geq \frac{3}{2}$. Since the function $x \mapsto x^r$ is convex, we have $\frac{1}{2}(a + x)^{-2r} + \frac{1}{2}(a - x)^{-2r} \geq (\frac{1}{2}(a + x)^{-2} + \frac{1}{2}(a - x)^{-2})^r$ and hence

$$2^{1-r}((a + x)^{-2} + (a - x)^{-2})^r - (a + x)^{-2r} - (a - x)^{-2r} \leq 0$$

with equality for $x = 0$. Therefore, it suffices to show that

$$(1 - 2^{1-r})((a + x)^{-2} + (a - x)^{-2})^r (1 - 4x^2)^{2r}$$

has a global maximum at $x = 0$. It is enough to show that $g(x) := ((a + x)^{-2} + (a - x)^{-2})(1 - 4x^2)^2$ is decreasing on $[0, \frac{1}{2}]$.

Computing the derivative of g we find

$$\begin{aligned} g'(x) &= -16x(1 - 4x^2)((a + x)^{-2} + (a - x)^{-2}) + (1 - 4x^2)^2(-2(a + x)^{-3} + 2(a - x)^{-3}) \\ &= (1 - 4x^2)(a + x)^{-3}(a - x)^{-3}k(x), \end{aligned}$$

where

$$\begin{aligned} k(x) &= -16x(a^2 - x^2)((a + x)^2 + (a - x)^2) + (1 - 4x^2)(2(a + x)^3 - 2(a - x)^3) \\ &= -16x \cdot 2(a^4 - x^4) + (1 - 4x^2) \cdot 4x \cdot (3a^2 + x^2) \\ &= 4x[-8(a^4 - x^4) + (1 - 4x^2)(3a^2 + x^2)] \\ &= 4x[4x^4 + (1 - 12a^2)x^2 + (3a^2 - 8a^4)]. \end{aligned}$$

Since $a > \sqrt{\frac{3}{8}}$, the function $p(y) := 4y^2 + (1 - 12a^2)y + (3a^2 - 8a^4)$ has a positive and a negative root. The sum of the two roots equals $\frac{12a^2 - 1}{4}$ and therefore the positive root is larger than $3a^2 - \frac{1}{4} \geq \frac{26}{4}$. It follows that p is negative on $[0, \frac{1}{4}]$ and hence $g'(x) = (1 - 4x^2)(a + x)^{-3}(a - x)^{-3} \cdot 4x \cdot p(x^2) \leq 0$ on $[0, \frac{1}{2}]$, which finishes the proof. \square

Added in proof. After this paper had been accepted for publication, Tom Koornwinder sent us the following interesting proof for the case that the parameter r in Proposition 2 is integral. With his kind permission we reproduce it here.

We consider $f_r(x)$ on $(0, 1)$. In terms of the Hurwitz zeta-function $\zeta(s, q)$ (see [6, Eq. 25.11.1]) we have

$$f_r(x) = \pi^{-2r} \sin^{2r}(\pi x)(\zeta(2r, x) + \zeta(2r, 1 - x)), \quad r = 1, 2, \dots$$

In terms of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ (see [6, Eq. 25.11.12]) this can be rewritten as

$$f_r(x) = \frac{\pi^{-2r} \sin^{2r}(\pi x)}{(2r-1)!} \left(\frac{d}{dx} \right)^{2r-1} (\psi(x) - \psi(1-x)).$$

Applying the reflection formula $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ (see [6, Eq. 5.5.4]) we obtain

$$f_r(x) = \frac{-\pi^{1-2r} \sin^{2r}(\pi x)}{(2r-1)!} \left(\frac{d}{dx} \right)^{2r-1} \cot(\pi x).$$

Substitution of $t = \pi x$ simplifies this expression to

$$\frac{(2r-1)!}{\sin^{2r} t} \cdot f_r(t/\pi) = - \left(\frac{d}{dt} \right)^{2r-1} \cot t.$$

Since $(d/dt) \cot t = -1/\sin^2 t$, we have $f_1(t/\pi) = 1$. Also, we obtain the following recursion relation:

$$\frac{(2r+1)!}{\sin^{2r+2} t} \cdot f_{r+1}(t/\pi) = - \left(\frac{d}{dt} \right)^{2r+1} \cot t = (2r-1)! \left(\frac{d}{dt} \right)^2 \frac{f_r(t/\pi)}{\sin^{2r} t}. \quad (10)$$

A small computation shows that

$$\begin{aligned} \left(\frac{d}{dt} \right)^2 \frac{f_r(t/\pi)}{\sin^{2r} t} &= \left(\frac{d}{dt} \right) \left[(\sin^{-2r} t) \left(\frac{d}{dt} \right) f_r(t/\pi) - 2r(\cos t)(\sin^{-2r-1} t) f_r(t/\pi) \right] \\ &= (\sin^{-2r} t) \left(\frac{d}{dt} \right)^2 f_r(t/\pi) - 4r(\cos t)(\sin^{-2r-1} t) \left(\frac{d}{dt} \right) f_r(t/\pi) \\ &\quad + (2r(2r+1) \cos^2 t \sin^{-2r-2} t + 2r \sin^{-2r} t) \cdot f_r(t/\pi). \end{aligned}$$

Hence, (10) implies that

$$\begin{aligned} \frac{(2r+1)!}{(2r-1)!} f_{r+1}(t/\pi) &= (\sin^2 t) \left(\frac{d}{dt} \right)^2 f_r(t/\pi) - 4r(\cos t)(\sin t) \left(\frac{d}{dt} \right) f_r(t/\pi) \\ &\quad + (2r(2r+1) \cos^2 t + 2r \sin^2 t) \cdot f_r(t/\pi). \end{aligned} \quad (11)$$

Set $y := \cos^2 t$ and $D := d/dy$. So $d/dt = -2(\sin t \cos t)D$ and

$$\begin{aligned} \left(\frac{d}{dt} \right)^2 &= \frac{d}{dt} (-2 \sin t \cos t) D \\ &= -2(\cos^2 t - \sin^2 t) D - (2 \sin t \cos t) \left(\frac{d}{dt} \right) D \\ &= -2(\cos^2 t - \sin^2 t) D + (4 \sin^2 t \cos^2 t) D^2 \\ &= (-4y + 2) D + 4y(1-y) D^2. \end{aligned}$$

Equation (11) can therefore be rewritten as

$$\begin{aligned} \frac{(2r+1)!}{(2r-1)!} f_{r+1}(t/\pi) &= [4y(1-y)^2 D^2 + ((8r-4)y+2)(1-y)D + 2r(2ry+1)] f_r(t/\pi) \\ &= [4y(r-yD)^2 + 8(r-yD)yD + 2yD + 2r + 4yD^2 + 2D] f_r(t/\pi). \end{aligned} \quad (12)$$

Observe that $(r-yD)y^k = (r-k)y^k$. Hence, if $p = p(y)$ is a polynomial of degree $n < r$ with nonnegative coefficients, then the same holds for $(r-yD)p$. The recursion (12) and the fact that $f_1(t/\pi) = 1$ now imply that $f_r(t/\pi)$ is a polynomial in y of degree $r-1$ with nonnegative

coefficients. The first few are given explicitly by

$$\begin{aligned} f_1(t/\pi) &= 1 \\ f_2(t/\pi) &= \frac{1}{3} + \frac{2}{3} \cos^2 t \\ f_3(t/\pi) &= \frac{2}{15} + \frac{11}{15} \cos^2 t + \frac{2}{15} \cos^4 t \\ f_4(t/\pi) &= \frac{17}{315} + \frac{4}{7} \cos^2 t + \frac{38}{105} \cos^4 t + \frac{4}{315} \cos^6 t \\ f_5(t/\pi) &= \frac{62}{2835} + \frac{1072}{2835} \cos^2 t + \frac{484}{945} \cos^4 t + \frac{247}{2835} \cos^6 t + \frac{2}{2835} \cos^8 t \end{aligned}$$

For integers r , Proposition 2 is an immediate consequence.

For half-integers $r = n + \frac{1}{2}$ one could observe that the identity

$$\psi^{(2n)}(x) = -(2n)! \sum_{m=0}^{\infty} \frac{1}{(x+m)^{2n+1}}$$

allows one to express the inequality of Proposition 2 in terms of the polygamma functions $\psi^{(2n)}$. We have not been able, however, to use this fact to give a simpler proof in that case.

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